SOLUTION OF THE GENERAL EQUATION OF THE FIFTH DEGREE.

TRANSLATED BY DR. A. B. NELSON, DANVILLE, KENTUCKY.

(*Continued from page* 14.)

§. 15.

Let there be given an equation of the 6th degree:

I. $\quad x^6 - 3bx^5 + Bx^4 - Cx^3 + Dx^2 - Ex + G = 0.$

I wish to construct its Capital.

$$x = (x_{,} + \tfrac{1}{2}b),\ \text{thus:}$$

$$\begin{array}{rcl}
x^6 & = & x_{,}^6 + 3bx_{,}^5 + \tfrac{15}{4}b^2x_{,}^4 + \tfrac{5}{2}b^3x_{,}^3 + \tfrac{15}{16}b^4x_{,}^2 + \tfrac{3}{16}b^5x_{,} + \tfrac{1}{64}b^6 \\
-3bx^5 & = & -3bx_{,}^5 - \tfrac{15}{4}b^2x_{,}^4 - \tfrac{15}{2}b^3x_{,}^3 - \tfrac{15}{4}b^4x_{,}^2 - \tfrac{15}{16}b^5x_{,} - \tfrac{3}{32}b^6 \\
Bx^4 & = & + Bx_{,}^4 + 2bBx_{,}^3 + \tfrac{3}{2}b^2Bx_{,}^2 + \tfrac{1}{2}b^3Bx_{,} + \tfrac{1}{16}b^4B \\
-Cx^3 & = & - Cx_{,}^3 - \tfrac{3}{2}bCx_{,}^2 - \tfrac{3}{4}b^2Cx_{,} - \tfrac{1}{8}b^3C \\
Dx^2 & = & + Dx_{,}^2 + bDx_{,} + \tfrac{1}{4}b^2D \\
-Ex & = & -Ex_{,} - \tfrac{1}{2}bE \\
G & = & + G.
\end{array}$$

Therefore:

$$\begin{array}{ll}
\text{II.} & x_{,}^6 + (-\tfrac{15}{4}b^2 + B)x_{,}^4 - (5b^3 - 2bB + C)x_{,}^3 \\
& + (-\tfrac{45}{16}b^4 + \tfrac{3}{2}b^2B - \tfrac{3}{2}bC + D)x_{,}^2 \\
& - (+\tfrac{3}{4}b^5 - \tfrac{1}{2}b^3B + \tfrac{3}{4}b^2C - bD + E)x_{,} \\
& + (-\tfrac{5}{64}b^6 + \tfrac{1}{16}b^4B - \tfrac{1}{8}b^3C + \tfrac{1}{4}b^2D - \tfrac{1}{2}bE + G) = 0.
\end{array}$$

§. 16.

In §. 10 I put $(yv+yt+vz+zu+ut) = m$ and found in §. 11 that there are 6 symmetrical halves of b whose sum $= 3b$, viz.:

$$\begin{array}{ll}
\text{I.} & (yv+yt+vz+zu+ut) \\
& (yz+yu+vz+vt+ut) \\
& (yv+yu+vt+zu+zt) \\
& (yz+yt+vu+vt+zu) \\
& (yv+yz+vu+zt+ut) \\
& (yu+yt+vz+vu+zt).
\end{array}$$

I use these six expressions as the unknown quantities of an equation of the 6th degree and construct the latter, viz.:

II. $x_{\prime}^{6}-Ax_{\prime}^{5}+Bx_{\prime}^{4}-Cx_{\prime}^{3}+Dx_{\prime}^{2}-Ex_{\prime}+G=0,$

which gives:

$$
\text{III.}\quad \begin{aligned}
A &= 3b,\\
B &= (3b^2-5d),\\
C &= (b^3-10bd),\\
D &= (-8b^2d+bc^2-25ce+15d^2),\\
E &= (-3b^3d+b^2c^2-25bce+15bd^2+\sqrt{P}^{*}),\\
G &= \frac{(-15b^2ce+2b^2d^2+125be^2+7bc^2d-50cde-2c^4+10d^3+b\sqrt{P})}{2}.
\end{aligned}
$$

If I now construct the capital of equation II., I find by the use of §. 15,

$$
\text{IV.}\quad (-\tfrac{15}{4}b^2+B)=\tfrac{1}{4}(-15b^2+12b^2-20d)=-\tfrac{1}{4}(3b^2+20d)
$$

$$
(5b^3-2bB+C)=(5b^3-6b^3+10d+b^3-10d)=0,
$$

$$
\begin{aligned}
(-\tfrac{45}{16}b^4+\tfrac{3}{2}b^2B-\tfrac{3}{2}bC+D) &= (-\tfrac{45}{16}b^4+\tfrac{9}{2}b^4-\tfrac{15}{2}b^2d-\tfrac{3}{2}b^4+15b^2d-8b^2d+bc^2\\
&\quad -25ce+15d^2)=\tfrac{1}{16}(3b^4-8b^2d+16bc^2-400ce+240d^2),
\end{aligned}
$$

$$
\begin{aligned}
(\tfrac{3}{4}b^5-\tfrac{1}{2}b^3B+\tfrac{3}{4}b^2C-bD+E) &= (\tfrac{3}{4}b^5-\tfrac{3}{2}b^5+\tfrac{5}{2}b^3d+\tfrac{3}{4}b^5-\tfrac{15}{2}b^3d+8b^3d-b^2c^2\\
&\quad +25bce-15bd^2-3b^3d+b^2c^2-25bce+15bd^2+\sqrt{P})\\
&= \sqrt{P}.
\end{aligned}
$$

$$
\begin{aligned}
&(-\tfrac{5}{64}b^6+\tfrac{1}{16}b^4B-\tfrac{1}{8}b^3C+\tfrac{1}{4}b^2D-\tfrac{1}{2}bE+G)=[-\tfrac{5}{64}b^6+\tfrac{1}{16}(3b^6-5b^4d)-\tfrac{1}{8}(b^6\\
&-10b^4d)+\tfrac{1}{4}(-8b^4d+b^3c^2-25b^2ce+15b^2d^2)+\tfrac{1}{2}(3b^4d-b^3c^2+25b^2ce-15b^2d^2-b\sqrt{P})\\
&+\tfrac{1}{2}(-15b^2ce+2b^2d^2+125be^2+7bc^2d-50cde-2c^4+10d^3+b\sqrt{P})]\\
&=\tfrac{1}{64}(-b^6+28b^4d-16b^3c^2-80b^2ce-176b^2d^2+4000be^2+224bc^2d-1600cde\\
&\qquad -64c^4+320d^3).
\end{aligned}
$$

Consequently

$$
\begin{aligned}
\text{V.}\quad & x_{\prime}^{6}-\tfrac{1}{4}(3b^2+20d)x_{\prime}^{4}+\tfrac{1}{16}(3b^4-8b^2d+16bc^2-400ce+240d^2)x_{\prime}^{2}-\sqrt{P}.x_{\prime}\\
&+\tfrac{1}{64}(-b^6+28b^4d-16b^3c^2-80b^2ce-176b^2d^2+4000be^2+224bc^2d-1600cde\\
&\qquad -64c^4+320d^3)=0.
\end{aligned}
$$

Whence it follows next that an equation of the 5th degree which has two equal unknown quantities is solvable algebraically; that is to say its unknown quantities can be represented by an algebraic function of b, c, d, e, because then the product of the differences of the unknown quantities has a factor $(y-y)=0$, and is thus itself $=0$. Therefore equation V can be resolved into a cubic with unknown quantities of quadratic form; consequently $x_{\prime}$ is found equal to m and hence the 5 unknown quantities of the given equation, $x^5+bx^3-cx^2+dx-e=0$, are known.

$^{*}\sqrt{P}=$ the product of the 10 differences in one direction. See §. 4.

But in the foregoing general form also the solution of equation V. is feasible. This is indeed an equation with its highest exponent 6, but in proper significance not an equation of the 6th degree* whose *general* solution formula is therefore not applicable here, since this (for the case in which the absolute term $=0$) must embrace in itself the possibility of a conversion into the formula for equations of the 5th degree, and therefore must contain radicals whose index is 5. The formula for m cannot have such radicals. In §. 8 it was shown that outside the outer radicals of the 5th degree, others could not generally occur in the irrational quantity of the formula of the 5th degree.

In its generality m indeed has 12 dimensions, but by combining the only quadratic expression $\sqrt{P}$ (see p. 1, preliminary remark) occurring among the coefficients, the irrational quantities involved will have only 6 dimensions, and only radicals with the indices 2 and 3 can appear in the same.

I assume that if in equation V. t and consequently e become $= 0$ (thus V. the fundamental equation of the 4th degree) all the elements which have the factor e, disappear.

I write equation V

VI. (1) $x_{,}^{6} - \frac{1}{4}Ax_{,}^{4} + \frac{1}{16}Bx_{,}^{2} - Dx_{,} + \frac{1}{64}C = 0.$

But in order to signify that I assume the element $e=0$, I write equation V

(2) $x_{,}^{6} - \frac{1}{4}\alpha x_{,}^{4} + \frac{1}{16}\beta x_{,}^{2} - \delta x_{,} + \frac{1}{64}\gamma = 0,$

and I have for this case:

VII. (1) $\alpha = (3b_{,}^{2}+20d_{,}),$

(2) $\beta = (3b_{,}^{4}-8b_{,}^{2}d_{,}+16b\,c_{,}^{2}+240d_{,}^{2}),$

(3) $\gamma = (-6b_{,}^{6}+28b_{,}^{4}d_{,}-16b_{,}^{3}c_{,}^{2}-176b_{,}^{2}d_{,}^{2}+224b_{,}c_{,}^{2}d_{,}-64c_{,}^{4}+320d_{,}^{3}),$

(4) $\delta = d_{,}\sqrt{(16b_{,}^{4}d_{,}+144b_{,}c_{,}^{2}d_{,}-128b_{,}^{2}d_{,}^{2}-4b_{,}^{3}c_{,}^{2}-27c_{,}^{4}+256d_{,}^{3})},$

which last expression I will also write $d_{,}\sqrt{[P(4)]}$, for the expression under the radical is the product of differences for the equation of the 4th degree.

From the foregoing equations (1) to (3) I am now able to represent the three coefficients $b_{,}$, $c_{,}$ and $d_{,}$ in terms of α, β, γ, and indeed by means of an equation of the 4th degree, viz., for $d_{,}$:

$$\text{VIII. (1)}\quad \left(\frac{\alpha-20d}{3}\right)^{4} - \frac{520}{352}\alpha\left(\frac{\alpha-20d}{3}\right)^{3} - \frac{170\beta-302\alpha^{2}}{352}\left(\frac{\alpha-20d}{3}\right)^{2} - \frac{82\alpha^{3}-130\alpha\beta-100\gamma}{352}\left(\frac{\alpha-20d}{3}\right) + \frac{(3\alpha^{2}-5\beta)^{2}}{352} = 0;$$

thus (2) $d_{,} = \text{function } \alpha\beta\gamma.$

---

*It has properties of the equation of the 2nd and of the equation of the 3d degree.

(3) $\delta = d_{,}^{2}P(4)$ signifies the determined relation of $b_{,,}$ $c_{,,}$ $d_{,}$ to one another. If I introduce into (3) the values of $b_{,,}$ $c_{,,}$ $d_{,}$ found from (1), I have:

$$(4)\ \left(\frac{a-20d_{,}}{3}\right)^4 - \left(\frac{404472a^2+575280\beta}{340704a}\right)\left(\frac{a-20d_{,}}{3}\right)^3$$
$$+\left(\frac{171986a^3+540930a\beta-453600\gamma}{340704a}\right)\left(\frac{a-20d_{,}}{3}\right)^2$$
$$-\left(\frac{27583a^4+217910a^2\beta-596700a\gamma}{340704a}\right)\left(\frac{a-20d_{,}}{3}\right)$$
$$+\left(\frac{859a^5+33840a^3\beta-73575a\beta^2-148500a\gamma+140800000}{340704a}\right)=0,$$

or (5) $\delta =$ function $\alpha\beta\gamma$. And by equating (4) and (1) I moreover depress the degree of the equation for $d_{,}$.

I assume that the values A, B, C, D [VI. (1)] may be introduced into the equation [from (5)].

IX. (1) Function $\alpha\beta\gamma\delta = 0$; thus

(2) Function $A.\ B.\ C.\ D = N$ having been formed, N must$=0$.

For if N had any value it must contain the factor e, since

Function $A.\ B.\ C.\ D = N$ immediately becomes

Function $\alpha\ \ \beta\ \ \gamma\ \ \delta = 0$, when I put $e = 0$.

But if N contain the factor $e_{,,}$ then the other member of the equation must have an equal factor. Hence there would arise a new equation;

(3) Function $b, c, d, e = e$. Whence I would be able to deduce

(4) Function $b, c, d \ = e$; but this is contradictory to the character of the equation:

$$x^5+bx^3-cx^2+dx-e=0,$$

which is *general*: one whose coefficients may be chosen arbitrarily and therefore must be independent of one another. From

(5) Function $A.\ B.\ C = D$ it follows that the unknown quantities of equation VI. (1), answering to V., may be represented by a function of $A.\ B.\ C.$

§. 17.

Let

I. $x^4+bx^2-cx+d=0.$

The 6 symmetrical halves of $b = yv+yz+yu+vz+vu+zu$, whose sum $=$ $3b$, are:

II. $(yv+vz+zu) = n$, and $n_{,} = (n - \frac{1}{2}b)$

$(yz+yu+vz)$

$(yv+yu+zu)$
$(yz+vu+zu)$
$(yv+yz+vu)$
$(yu+vz+vu)$

If I represent n (which for equations of the 4th degree is supposed to be known) by means of the coefficients b, c, d, provided

$$f=\sqrt[3]{\tfrac{1}{2}\left\{(2b^3-72bd+27c^2)+\sqrt{[(2b^3-72bd+27c^2)^2-(b^2+12d)^3]}\right\}},$$
$$f'=\tfrac{1}{2}f(-1+\sqrt{-3}),\ f''=\tfrac{1}{2}f(-1-\sqrt{-3}),$$

I have

$$\text{III. } n_{,}=\frac{2\left(f+\dfrac{b^2+12d}{f}\right)+\left(f'+\dfrac{b^2+12d}{f'}\right)+\sqrt{-12d+3\left(b+f'+\dfrac{b^2+12d}{f'}\right)^2}}{6}.$$

I now assume farther that from equation § 16. VI. (1)

IV. $m_{,}=(m-\frac{1}{2}b)=$ function ABC (without D) may be discovered, which is possible, since $m_{,}$ is a function of the coefficients of its equation and D is a function of ABC; thus if I put $e=0$, IV will be at once converted into III. But as it is impossible that any further change can occur in the position of the radicals in the formula for $m_{,}$, than that one of them shall entirely vanish or an indicated root appear as actually accomplished, formula III. is in reality a symbol of the expression for $m_{,}$.

If I find from §. 16. VIII. the values of $b_{,}$, $c_{,}$, $d_{,}$ according to α, β, γ and introduce these results into III., there thus arises another formula for $n_{,}$, which I designate

V. $n_{,}=$ Function $\alpha\beta\gamma$: similarly I am able to represent the 5 other values of $n_{,}$.

If I now construct from these 6 values of the unknown quantities, equatton §. 16. VI. (2) with respect to its coefficients, $\alpha\,\beta\,\gamma$ will occupy the same position in the result which they now have in formula §. 16. VI. (2).

If I therefore exchange in formula V. α with A, β with B, and finally γ with C, there arise 6 expressions $m_{,}$ etc., which, treated as above described for $n_{,}$ and its correspondents must give the formula VI. (1) (with previous exclusion of D).

But, moreover, the coefficient D must appear in the position which in formula §. 16. VI. (2) the coefficient δ occupies; for δ was derived from D by putting $e=0$ in the equivalent function $A.\ B.\ C.$ Thus, in the next place, all the terms in δ and D which do not have the element e, must be identical.

And it follws from this mode of derivation also that the position of the extra elements in D and δ is generally the same, for by having taken the

above described collocation of the coefficients δ from $n_{,}$ and its 5 correspon-ding values, I completed the operation indicated in the equation $\delta = \alpha\beta\gamma$. If I now put together D and $m_{,}$ etc., the extra symbols in A. B. C must pass through the same algebraic processes of transformation as was necessary with $n_{,}$ etc. in its formation from δ, and the result must consequently be D.*

§. 18.

From the investigations in §§. 16 and 17 the possibility of the algebraic representation of $m_{,} = m - \frac{1}{2}b$ results, as follows:

From the equation §. 16. VIII. (which, as already mentioned, can be depressed, and indeed to the first degree for d) I find the values of b, c, d in terms of α, β, γ [§. 16. VI. (2)], exchange them with A, B, and C respectively, and introduce the result thus found into the formula §. 17. III. Hence $m_{,}$ becomes known.

From $m_{,}$ I have $m = (m_{,}-\frac{1}{2}b)$, and hence I am able (§. 14.) to develop y, with which the solution of the equation of the 5th degree is complete.

I had indeed striven earnestly not to go beyond the solution of the equation of the 5th degree, but my material crowds its limits, and one observation breaks over the rule.

The direct proof that the exchanges of the coefficients of VI. (1), made in the formula for the unknown quantities of VI· (2), is well grounded, is connected with the basis of the problem of the *general solution*, viz., to establish the relation between $(n—1)$ constants (§. 5) of an equation of the n^{th} degree and the $(n-1)$ coefficients of a *complete* equation of the $(n-1)^{\text{th}}$ degree.

There are thus in the present case the four coefficients of the complete equation of the 4th degree to be represented in the radicals of formula III., for it is possible also, without reference to particular properties of the products of differences, to construct from these 4 elements and the 4 coefficients of equation VI. (1) four equations of the 4th degree for determining $a_{,,}$, $b_{,,}$, $c_{,,}$ and $d_{,,}$, and consequently, to discover the coefficients.

[The translator of the foregoing solution is conscious of having executed his task imperfectly. There are two or three expressions especially which, either from the difficulty of the thought or the obscurity of its expression in the original, he fears he has not adequately rendered. It is hoped, however, that in the solntion as given the reasoning as a whole will be intelligible to all.—*Translator.*]

---

*Or: Let the product of differences for the equation of the 5th degree be $P(5) =$ function A. B. C. By putting $e=0$ I exchange A with α, B with β, C with γ, and there arises: $P(4) =$ function α. β. γ. Therefore, if in the last expression for α. β. γ I substitute A. B. C respectively, $P(5)$ is reproduced.